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# SOME APPLICATIONS OF DENSITY THEOREM

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## ABSTRACT

*We know that, both rationals and irrationals are dense in ℝ. Therefore, ℝ has a dense subset, whose complement is also dense in ℝ. In this paper, I am trying to construct so many counter examples by using this idea, such that*

- *Construction of a function which is discontinuous everywhere.*
- *Construction of a function which is continuous exactly at one point (Or at finitely many points).*
- *Construction of a function which is differentiable at exactly one point (Or at finitely many points).*

*I am presenting all these constructions as an application of Density theorem.*

***KEYWORDS:*** *Application of Density Theorem, Sequence of Rationals*

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## INTRODUCTION

### Preliminaries & Definitions

### Density Theorem

If x and y are any real number with $x < y$, then, there exists a rational number $r \in Q$ such that $x < r < y$.

### Corollary

If x and y are any real number with $x < y$, then there exists an irrational number $i$ such that $x < i < y$.

### Sequential Criterion for Continuity

A function f: A→R is continuous at the point c in A, if for every sequence $(x_n)$ in A that converges to c, the sequence $(f(x_n))$ converges to f(c).

### Applications of Density Theorem

### Construction of a Rational Sequence which Converges to any x∈ℝ

If x is a real number, then there exists a sequence of rationals that converges to $x$.

### Proof

Let $x$ be a real number.

We know, $x - \frac{1}{n} < x + \frac{1}{n}\ \forall n \in \mathbb{N}$

∴ By Density theorem, there exist a rational number $r_n$ such that $x - \frac{1}{n} < r_n < x + \frac{1}{n}\ \forall n$

i.e, $|r_n - x| < \frac{1}{n}\ \forall n$

as $n \to \infty$, $(\frac{1}{n}) \to 0$

Which shows that $(r_n) \to x$.

i.e., $\lim(r_n) = x$.

∴ $(r_n)$ is a sequence of rationals that converges to $x$.

Note: In a similar way, we can discuss the case of irrationals also.

## Construction of a Function which is Discontinuous every where

$$f(x) = \begin{cases} 1 & if\ x\ rational \\ 0 & if\ x\ irrational \end{cases}$$

## Proof

Let x be a rational number.

Then, $f(x)=1$.

By 2.1, there exists a sequence of irrationals $(i_n) \to x$.

Suppose f is continuous.

By Sequential criterion of continuity,

$(f(i_n)) \to f(x)$

i.e., $\lim(f(i_n)) = f(x)$

i.e., $\lim(\ f(i_1), f(i_2), \ldots\ldots\ldots) = 1$

$\lim(0,0,0,\ldots..) = 1$

i.e., 0=1, this is a contradiction.

∴ f is discontinuous at $x$.

i.e., f is discontinuous at all rational points.

Similarly, we can discuss the case when x is irrational

i.e., f is discontinuous at all irrational points.

Hence, f is discontinuous everywhere on $\mathbb{R}$.

This function is known as "Dirichlet Unit function".

## Construction of a Function Which is Continuous Exactly at One Point

Consider the function,

$$f(x)=\begin{cases} x & \text{if } x \text{ rational} \\ -x & \text{if } x \text{ irrational} \end{cases}$$

## Proof

Let r be a rational number and suppose that f is continuous at r.

Let $(i_n)$ be the sequence of irrationals that converges to r.

i.e., $(i_n) \to r$.

Since f is continuous at r, by Sequential Criterion,

$\lim(f(i_n))=f(r)$

i.e., $\lim (f(i_n))=r$

$\lim(-i_n)=r$

$\lim(i_n)=r$

$\lim(i_n)=-r$

$r=-r$

$2r=0$

$r=0$

∴ f(x) is discontinuous everywhere expect at r=0.

Now,

Let i be the irrational number. Then f(i)=-i.

Let $(r_n)$ be the sequence of rationals that converges to i.

i.e., $\lim(r_n)=i$

Suppose f is continuous at i.

By Sequential Criterion of continuity,

$\lim (f(r_n))=f(i)$

i.e., $\lim(f(r_n))=-i$

$\lim(r_n)=-i$

$i=-i$

$2i=0$

i=0, a contradiction.

∴ f(x) is discontinuous at each irrationals.

Hence, f(x) is discontinuous everywhere on ℝ except possibly at x=0.

Now, we check whether this function is continuous at x=0.

We have $-x \leq f(x) \leq x \; \forall x$

$\lim_{x\to 0} -x = \lim_{x\to 0} x = 0$

∴ By Sandwich theorem,

$\lim_{x\to 0} f(x) = 0$

f(0)=0

$\therefore \lim_{x\to 0} f(x) = f(0)$

Which implies that f is continuous exactly at x=0.

**Note:**

This function is not differentiable at x=0.

It is clear that, this function is a sharp sandwich between |x| and -|x|.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

Here $f(0)=0$

$$\therefore f'(0) = \lim_{x\to 0} \frac{f(x)}{x}$$

$$\frac{f(x)}{x} = \begin{cases} \frac{x}{x} \; if \; x \; rational \; when \; x \neq 0 \\ \frac{-x}{x} if \; x \; irraional \; when \; x \neq 0 \end{cases}$$

$$= \begin{cases} 1 \; if \; x \; rational \; x \neq 0 \\ -1 \; if \; x \; irrational \end{cases}$$

$\therefore \lim_{x\to 0} \frac{f(x)}{x}$ does not exist here.

**Construction of a Function Which is Differentiable at Exactly One Point**

Now we consider the function

$$f(x)=\begin{cases} x^2 \text{ if } x \text{ rational} \\ -x^2 \text{ if } x \text{ irrational} \end{cases}$$

Here, we are using smooth sandwitch between $-x^2$ and $x^2$

**Proof**

Let r be a rational number and suppose that f is continuous at r.

Let $(i_n)$ be the sequence of irrationals that converges to r.

i.e., $(i_n) \rightarrow r$

Since f is continuous, by Sequential Criterion

$\lim(f(i_n))=f(r)$.

i.e., $\lim (f(i_n))=r^2$

$\lim(-i_n^2)=r^2$

$\lim(i_n^2)=r^2$

$\lim(i_n^2)= -r^2$

$r^2=-r^2$

$2r^{2=}0$

$r^2=0$

$r=0$

∴ f is discontinuous at all rationals except possibly at zero.

Similarly, we can discuss the case of irrationals also

∴ f is discontinuous at all irrationals.

Now,

We check whether this function is continuous at x=0

We have,

$-x^2 \leq f(x) \leq x^2 \; \forall \; x$

$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$

∴ By Sandwich theorem,

$\lim_{x\to 0} f(x) = 0$

$f(0)=0$

$\therefore \lim_{x\to 0} f(x) = f(0)$

Which implies that $f$ is continuous exactly at x=0.

We know that differentiability implies continuity. The contra-positive is discontinuity implies non-differentiability.

Hence, this function is not differentiable everywhere, except possibly at zero.

Now, we check differentiability at x=0.

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)}{x}$$

$$\frac{f(x)}{x} = \begin{cases} \frac{x^2}{x} \; if\ x\ rational\ when\ x \neq 0 \\ \frac{-x^2}{x} \; if\ x\ irraional\ when\ x \neq 0 \end{cases}$$

$$\frac{f(x)}{x} = \begin{cases} x \;\; if\ x\ rational \\ -x\ if\ x\ irrational \end{cases}$$

$$\therefore \lim_{x\to 0}\frac{f(x)}{x} = 0$$

$\therefore f'(0)=0$

Hence, $f'(0)$ exists.

Which shows that f is differentiable at x=0.

## CONCLUSIONS

Conclusions are

- There exists a sequence of rationals (irrationals) that converges to any $x \in \mathbb{R}$.
- There exists a function which is discontinuous everywhere.
- There exists a function which is continuous exactly at one point.
- There exists a function which is differentiable at exactly one point.

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